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Uniform or Mean Convergence of Hermite–Fejér Interpolation of Higher Order for Freud Weights

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In this paper we show the uniform or mean convergence of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with the typical Freud weight. © 1999 Academic Press

1. INTRODUCTION

Let $W^2(x) = \exp(-x^m)$, where $m = 2, 4, 6, \ldots$, be the typical Freud weight, and let the orthonormal polynomials $p_n(W^2; x)$ with the weight $W^2(x)$ be defined by the relation

$$\int_{-\infty}^{\infty} p_i(x)\, p_j(x)\, W^2(x)\, dx = \delta_{ij}, \qquad i, j = 0, 1, 2, \ldots, \tag{1.1}$$

where $p_n(x) = p_n(W^2: x) = \gamma_n x^n + \cdots$ with $\gamma_n > 0$. We denote the zeros of $p_n(x)$ by x_{kn}, $k = 1, 2, \ldots, n$, where $x_{1n} > x_{2n} > \cdots > x_{nn}$. Let v be a positive integer. For an arbitrary real valued continuous function $f \in C(R)$, the unique Hermite–Fejér interpolation polynomial $L_n(v, f; x) \in \Pi_{vn-1}$ of order v based at $\{x_{kn}\}$ is defined by

$$\begin{aligned} L_n(v, f; x_{kn}) &= f(x_{kn}), && k = 1, 2, \ldots, n, \\ L_n^{(r)}(v, f; x_{kn}) &= 0, && k = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, v-1, \end{aligned} \tag{1.2}$$

where Π_n is the set of all algebraic polynomials of degree $\leqslant n$. The interpolation polynomial $L_n(v, f; x)$ is written in the form

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$$L_n(v, f; x) = \sum_{k=1}^{n} f(x_{kn})\, h_{kn}(v; x), \qquad n = 1, 2, 3, ..., \tag{1.3}$$

where the polynomial $h_{kn}(v; x) \in \Pi_{vn-1}$ satisfies

$$\begin{aligned} h_{kn}(v; x_{pn}) &= \delta_{pk}, \qquad p = 1, 2, ..., n, \\ h_{kn}^{(r)}(v; x_{pn}) &= 0, \qquad p = 1, 2, ..., n, \quad r = 1, 2, ..., v-1. \end{aligned} \tag{1.4}$$

An explicit form of $h_{kn}(v; x)$ is

$$h_{kn}(v; x) = \ell_{kn}^{v}(x) \sum_{j=0}^{v-1} e_{jk}(x - x_{kn})^j, \qquad k = 1, 2, ..., n, \tag{1.5}$$

where $\ell_{kn}(x)$ are the Lagrange fundamental polynomials of degree exactly $n-1$, that is, with $\omega(x) = \Pi_{k=1}^{n} (x - x_{kn})$

$$\ell_{kn}(x) = \omega(x)/\{(x - x_{kn})\, \omega'(x_{kn})\}, \qquad k = 1, 2, ..., n. \tag{1.6}$$

and the coefficients e_{jk} can be calculated by (1.4) (see [3]). We see that $L_n(f; x) = L_n(1, f; x)$ is the Lagrange interpolation polynomial and $L_n(2, f; x)$ is the ordinary Hermite–Fejér interpolation polynomial. In [3] we showed the following.

THEOREM 1 [3, Corollary 1]. *Let $f \in C(R)$ be a uniformly continuous function on R. Then, for every $M > 0$, the sequence of Hermite–Fejér interpolation polynomials of even order v converges uniformly to f in the interval $[-M, M]$, that is,*

$$\lim_{n\to\infty} \max_{-M \leqslant x \leqslant M} |L_n(v, f; x) - f(x)| = 0.$$

THEOREM 2 [3, Corollary 2]. *Let v be an odd integer. For a and b with $a < b$, there exists $f \in C(R)$ such that*

$$\limsup_{n\to\infty} \max_{a \leqslant x \leqslant b} |L_n(v, f; x)| = \infty.$$

Recently, for the Lagrange interpolation polynomial $L_n(f; x)$, Lubinsky and Matjila [6] obtained the following nice result. Let $W_\beta^2(x) = \exp(-|x|^\beta)$, $\beta > 1$, be a Freud weight, and let $L_n(f; x)$ be the Lagrange interpolation polynomial based at the zeros $\{x_{kn}\}$ of the orthonormal polynomial $p_n(W_\beta; x)$. Let $1 < p < \infty$. For $\alpha \in R$ we define

$$\langle\!\langle \alpha \rangle\!\rangle = \min\{1, \alpha\}. \tag{1.7}$$

THEOREM 3 [6, Theorem 1.1]. *Let $\Delta \in R$ and $\alpha > 0$. Then, for*

$$\lim_{n\to\infty} \|(1 + |x|)^{-\Delta}\, W_\beta(x)\{L_n(f; x) - f(x)\}\|_{L_p(R)} = 0,$$

to hold for every continuous function $f\colon R \to R$ satisfying

$$\lim_{|x| \to \infty} (1+|x|)^{\alpha} W_{\beta}(x)\, |f(x)| = 0,$$

it is necessary and sufficient that

$$\begin{aligned} &\Delta > 1/p - \langle\!\langle \alpha \rangle\!\rangle && \text{if} \quad 1 < p \leqslant 4; \\ &\Delta > 1/p - \langle\!\langle \alpha \rangle\!\rangle + (\beta/6)(1-4/p) && \text{if} \quad p > 4 \quad \text{and} \quad \alpha = 1; \\ &\Delta \geqslant 1/p - \langle\!\langle \alpha \rangle\!\rangle + (\beta/6)(1-4/p) && \text{if} \quad p > 4 \quad \text{and} \quad \alpha \neq 1. \end{aligned}$$

Our purpose of this paper is to extend Theorem 3 to the Hermite–Fejér interpolation polynomials $L_n(v, f; x)$ based at the zeros $\{x_{kn}\}$ of the orthonormal polynomial $p_n(W^2; x)$ defined by (1.1). Let $\alpha, \Delta \in R$, and let us define

$$\langle x \rangle = \begin{cases} 1, & x < 1, \\ x, & x \geqslant 1. \end{cases} \tag{1.8}$$

First, we prove a uniform convergence theorem.

THEOREM 4. *Let $v = 2, 4, 6, \ldots$, and let $\alpha + \langle vm/6 \rangle - vm/6 \geqslant 0$. We fix an arbitrary constant $1 > \eta \geqslant 0$. If*

$$\Delta + (vm/6) \geqslant 0, \qquad \Delta + \langle\!\langle \alpha + \langle vm/6 \rangle - vm/6 \rangle\!\rangle \geqslant 0,$$

then, for every continuous function $f\colon R \to R$ satisfying

$$\lim_{|x| \to \infty} (1+|x|)^{\alpha + m - \eta + \langle vm/6 \rangle} W^{v}(x)\, |f(x)| = 0, \tag{1.9}$$

we have

$$\lim_{n \to \infty} \|(1+|x|)^{-(\Delta + vm/6)} W^{v}(x) \{L_n(v, f; x) - f(x)\}\|_{C(R)} = 0. \tag{1.10}$$

Remark. Let $v = 2, 4, 6, \ldots$, and fix an arbitrary constant $1 > \eta \geqslant 0$. If $m = v = 2$ and $\alpha + 1/3 \geqslant 0$, then for $\Delta + \min\{2/3, \langle\!\langle \alpha + 1/3 \rangle\!\rangle\} \geqslant 0$ we have for every continuous function $f\colon R \to R$ satisfying

$$\lim_{|x| \to \infty} (1+|x|)^{\alpha + 3 - \eta} W^{v}(x)\, |f(x)| = 0,$$

the estimation

$$\lim_{n \to \infty} \|(1+|x|)^{-(\Delta + 2/3)} W^{v}(x) \{L_n(v, f; x) - f(x)\}\|_{C(R)} = 0.$$

If $m\nu \neq 4$ and $\alpha \geqslant 0$, then for $\Delta + \langle\!\langle \alpha \rangle\!\rangle \geqslant 0$ we have for every continuous function $f\colon R \to R$ satisfying

$$\lim_{|x| \to \infty} (1+|x|)^{\alpha + (1+\nu/6)\, m - \eta}\, W^{\nu}(x)\, |f(x)| = 0,$$

the estimation

$$\lim_{n \to \infty} \|(1+|x|)^{-(\Delta + \nu m/6)}\, W^{\nu}(x) \{L_n(\nu, f; x) - f(x)\}\|_{C(R)} = 0.$$

The following are the analogues of Theorem 3.

THEOREM 5. *Let* $\nu = 2, 3, 4, ...,$ $1 < p < \infty$, *and* $\alpha > 0$. *Assume that*

$$\Delta > 1/p \qquad \text{if} \quad 1 < p \leqslant 4/\nu \quad (\nu < 4); \tag{1.11}$$

$$\Delta > 1/p \qquad \text{if} \quad (m/6)(\nu - 4/p) \leqslant \langle\!\langle \alpha \rangle\!\rangle, \quad p > 4/\nu; \tag{1.12}$$

$$\Delta \geqslant 1/p - \langle\!\langle \alpha \rangle\!\rangle + (m/6)(\nu - 4/p) \qquad \text{if} \quad (m/6)(\nu - 4/p) > \langle\!\langle \alpha \rangle\!\rangle, \quad p > 4/\nu. \tag{1.13}$$

(*Here, if* $4 \leqslant \nu$ *we omit* (1.11), *and we set* $p > 1$ *for* (1.12) *or* (1.13).) *Then, for every continuous function* $f\colon R \to R$ *satisfying*

$$\lim_{|x| \to \infty} (1+|x|)^{\alpha + (\nu-1)\, m/6}\, W^{\nu}(x)\, |f(x)| = 0, \tag{1.14}$$

we have

$$\lim_{n \to \infty} \|(1+|x|)^{-\Delta}\, W^{\nu}(x) \{L_n(\nu, f; x) - f(x)\}\|_{L_p(R)} = 0. \tag{1.15}$$

THEOREM 6. *let* $\nu = 3, 5, 7, ...,$ $1 < p < \infty$, *and* $\alpha > 0$. *Assume that for every continuous function* $f\colon R \to R$ *satisfying* (1.14) *we have* (1.15). *Then, the following inequalities hold.*

$$\Delta > 1/p - \langle\!\langle \alpha + (\nu - 1)\, m/6 \rangle\!\rangle \qquad \text{if} \quad 1 < p \leqslant 4/\nu \quad (\nu < 4); \tag{1.16}$$

$$\Delta > 1/p - \langle\!\langle \alpha + (\nu - 1)\, m/6 \rangle\!\rangle + (m/6)(\nu - 4/p) \qquad \text{if} \quad p > 4/\nu \quad \text{and} \quad \alpha + (\nu - 1)\, m/6 = 1; \tag{1.17}$$

$$\Delta \geqslant 1/p - \langle\!\langle \alpha + (\nu - 1)\, m/6 \rangle\!\rangle + (m/6)(\nu - 4/p) \qquad \text{if} \quad p > 4/\nu \quad \text{and} \quad \alpha + (\nu - 1)\, m/6 \neq 1. \tag{1.18}$$

If we consider the case of $v = 3, 5, 7, \ldots$, then we have the following

COROLLARY 7. *Let $v = 3, 5, 7, \ldots$, $\alpha \geqslant 1$, and let $(m/6)(v - 4/p) > 1$. Then, for (1.15) to hold for every continuous function f: $R \to R$ satisfying (1.14), it is necessary and sufficient that*

$$\Delta \geqslant 1/p - 1 + (m/6)(v - 4/p).$$

If $v = 3$, then we suppose $p > 4/3$.

2. PRELIMINARIES

The Hermite–Fejér interpolation polynomial $L_n(v, f; x)$ is defined by (1.2) and (1.3). The Lagrange fundamental polynomials $\ell_{kn}(x)$, $k = 1, 2, \ldots, n$, of degree exactly $n-1$ are defined by (1.6), and the fundamental polynomials $h_{kn}(v; x)$, $k = 1, 2, \ldots, n$, of $L_n(v, f; x)$ are defined by (1.5) with (1.4). For $u > 0$, the uth Mhaskar–Rahmanov–Saff number $a_u = a_u(w)$ is the positive root of the equation

$$u = (m/\pi)(a_u)^m \int_0^1 t^m (1 - t^2)^{-1/2}\, dt$$
$$= (m/2)\{(m-1)!!/m!!\}(a_u)^m. \tag{2.1}$$

Let γ_n be the leading coefficient of $p_n(x) = \gamma_n x^n + \cdots$, and we set $b_n = \gamma_{n-1}/\gamma_n$. Furthermore, we also use the number $q_n = (2n/m)^{1/m}$. Then, we see that

$$x_{1n} \sim a_n \sim b_n \sim q_n \sim n^{1/m} \tag{2.2}$$

as $n \to \infty$ (see (2.1), (2.3), and [5, (12.26)]), where for the positive functions $b(u)$ and $c(u)$, $b(u) \sim c(u)$ remarks that there exist $C_1, C_2 > 0$ independent of u such that $C_1 \leqslant b(u)/c(u) \leqslant C_2$.

We need some fundamental lemmas. Let C be a positive constant independent of k and n. First, we denote the useful lemmas from [6].

LEMMA 2.1 [6, Theorem 2.1]. (a) *For $n \geqslant 1$,*

$$|(x_{1n}/a_n) - 1| \leqslant C n^{-2/3}, \tag{2.3}$$

and uniformly for $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$,

$$x_{k-1,n} - x_{k+1,n} \sim (a_n/n)(\max\{n^{-2/3}, 1 - |x_{kn}|/a_n\})^{-1/2}. \tag{2.4}$$

(b) *For* $n \geqslant 1$,

$$\sup_{x \in R} |1 - |x|/a_n|^{1/4} W(x) |p_n(x)| \sim a_n^{-1/2}. \tag{2.5}$$

and

$$\sup_{x \in R} W(x) |p_n(x)| \sim n^{1/6} a_n^{-1/2}. \tag{2.6}$$

(c) *Uniformly for* $n \geqslant 1$ *and* $1 \leqslant k \leqslant n$,

$$W(x_{kn}) |p_n'(x_{kn})| \sim n a_n^{-3/2} (\max\{n^{-2/3}, 1 - |x_{kn}|/a_n\})^{1/4} \qquad (\textit{by } [5, (1.19)]). \tag{2.7}$$

(d) *Let* $0 < p \leqslant \infty$. *There exists* $C > 0$ *such that for* $n \geqslant 1$ *and* $P \in \Pi_n$,

$$\| WP \|_{L_p(R)} \leqslant C \| WP \|_{L_p[-a_n, a_n]}. \tag{2.8}$$

Lemma 2.2 [6, Theorem 2.2]. (a) *Given* $0 < p < \infty$, *we have for* $n \geqslant 1$,

$$\| W p_n \|_{L_p(R)} \sim a_n^{1/p - 1/2} \times \begin{cases} 1, & 0 < p < 4 \\ \{\log(1+n)\}^{1/4}, & p = 4 \\ n^{(1/6)(1-4/p)}, & p > 4. \end{cases} \tag{2.9}$$

(b) *Uniformly for* $n \geqslant 1$, $1 \leqslant k \leqslant n$, *and* $x \in R$,

$$|\ell_{kn}(x)| \sim (a_n^{3/2}/n) W(x_{kn}) (\max\{n^{-2/3}, 1 - |x_{kn}|/a_n\})^{-1/4} \times |p_n(x)/(x - x_{kn})| \qquad (\textit{by } (1.6), (2.7)). \tag{2.10}$$

(c) *Uniformly for* $n \geqslant 1$, $1 \leqslant k \leqslant n$, *and* $x \in R$,

$$|W^{-1}(x_{kn}) W(x) \ell_{kn}(x)| \leqslant C. \tag{2.11}$$

Lemma 2.3 [3, Lemma 6, Lemma 14 (4.16)]. *Let* e_{jk} *be the coefficient of* (1.5). *Then, by* (2.2) *we have*

$$|e_{jk}| \leqslant C(n/a_n)^j, \qquad j = 0, 1, ..., \nu - 1, \quad k = 1, 2, ..., n, \tag{2.12}$$

especially, for odd number j

$$|e_{jk}| \leqslant C M_n(x_{kn})(n/a_n)^{j-1}, \qquad k = 1, 2, ..., n, \tag{2.13}$$

where

$$M_n(x_{kn}) = a_n^{-2} |x_{kn}| + |x_{kn}|^{m-1}, \qquad k = 1, 2, ..., n. \tag{2.14}$$

Furthermore, we need a certain generalized Hermite–Fejér interpolation polynomial of (ℓ, ν)-order, $\ell = 0, 1, ..., \nu - 1$ (cf. [4]). For $f \in C^{(\ell)}(R)$ we define $L_n(\ell, \nu, f; x) \in \Pi_{\nu n-1}$ by

$$L_n^{(r)}(\ell, \nu, f; x_{kn}) = f^{(r)}(x_{kn}), \qquad r = 0, 1, ..., \ell,$$
$$L_n^{(r)}(\ell, \nu, f; x_{kn}) = 0, \qquad r = \ell+1, \ell+2, ..., \nu-1, \quad k = 1, 2, ..., n.$$

The polynomial $L_n(\ell, \nu, f; x)$ is written in the form

$$L_n(\ell, \nu, f; x) = \sum_{k=1}^{n} \sum_{s=0}^{\ell} f^{(s)}(x_{kn})\, h_{skn}(\nu; x), \qquad n = 1, 2, 3, ...,$$

where for the polynomial $h_{skn}(\nu; x) \in \Pi_{\nu n-1}$

$$h_{skn}^{(j)}(\nu; x_{pn}) = \delta_{js}\, \delta_{pk}, \qquad s = 0, 1, ..., \ell,\ j = s, s+1, ..., \nu-1,\ p, k = 1, 2, ..., n. \tag{2.15}$$

An explicit form of $h_{skn}(\nu; x)$ is

$$h_{skn}(\nu; x) = \ell_{kn}^{\nu}(x) \sum_{j=s}^{\nu-1} e_{jsk}(x - x_{kn})^j, \qquad k = 1, 2, ..., n, \tag{2.16}$$

where the $\ell_{kn}(x)$ are the Lagrange fundamental polynomials.

We see that $L_n(0, \nu, f; x)$ is the Hermite–Fejér interpolation polynomial $L_n(\nu, f; x)$ of order ν, and $L_n(\nu-1, \nu, f; x)$ preserves any polynomial $P \in \Pi_{\nu n-1}$, that is,

$$L_n(\nu-1, \nu, P; x) = P(x), \qquad x \in R. \tag{2.17}$$

From (2.15) we obtain the following.

LEMMA 2.4 [4, Lemma 3]. *For the coefficients e_{jsk} we have*

$$|e_{jsk}| \leqslant C(n/a_n)^{j-s}, \qquad s = 0, 1, ..., \ell, \quad j = s, s+1, ..., \nu-1, \quad k = 1, 2, ..., n. \tag{2.18}$$

From (2.4) there exists a positive constant δ such that

$$\delta a_n/n \leqslant x_{j-1,n} - x_{j+1,n}, \qquad j = 1, 2, ..., n, \tag{2.19}$$

where

$$x_{0n} = x_{1n}(1 + n^{-2/3}), \qquad x_{n+1,n} = x_{nn}(1 - n^{-2/3}) \qquad \text{(cf. (2.3))}.$$

Therefore, if we set

$$x_{kn} - x = t(k, x)\, \delta a_n / n, \qquad k = 0, 1, ..., n+1, \tag{2.20}$$

then we see that

$$t(n+1, x) < t(n, x) < \cdots < t(1, x) < t(0, x),$$

and

$$t(j-1, x) - t(j+1, x) \geqslant 1, \qquad j = 1, 2, ..., n.$$

3. PROOF OF THEOREM 4

Throughout this section we assume that $\alpha + \langle \nu m/6 \rangle - \nu m/6 \geqslant 0$, $\Delta + \nu m/6 \geqslant 0$, and $\Delta + \langle\!\langle \alpha + \langle \nu m/6 \rangle - \nu m/6 \rangle\!\rangle \geqslant 0$, where $\langle\!\langle \cdot \rangle\!\rangle$ and $\langle \cdot \rangle$ are defined by (1.7) and (1.8), respectively.

LEMMA 3.1. *Let $\nu = 2, 4, 6, ...,$ and $\varepsilon > 0$, $1 > \eta \geqslant 0$. If $g \in C(R)$ satisfies*

$$(1 + |x|)^{\alpha + m - \eta + \langle \nu m/6 \rangle} W^\nu(x)\, |g(x)| < \varepsilon, \qquad x \in R, \tag{3.1}$$

then we have

$$\sum(x) = (1 + |x|)^{-(\Delta + \nu m/6)} W^\nu(x) \sum_{k=1}^{n} |g(x_{kn})\, h_{kn}(x)| < C\varepsilon, \qquad x \in R, \tag{3.2}$$

where C is a positive constant independent of n and ε.

LEMMA 3.2. *Let $\nu = 1, 2, 3, ...,$ and $1 > \eta \geqslant 0$. If $g \in C(R)$ satisfies that for a positive constant $M(g)$,*

$$(1 + |x|)^{\alpha + m - \eta + \langle \nu m/6 \rangle} W^\nu(x)\, |g(x)| < M(g), \qquad x \in R,$$

where $M(g)$ may depend on g, then, for every $x \in R$ we have

$$\sum(x) = (1 + |x|)^{-(\Delta + \nu m/6)} W^\nu(x) \sum_{k=1}^{n} |g(x_{kn})\, h^*_{kn}(x)| < CM(g) \log(1+n),$$

where

$$h^*_{kn}(x) = |\ell^{\nu}_{kn}(x)| \sum_{j=0}^{\nu-1} (n/a_n)^j\, |x - x_{kn}|^j, \tag{3.3}$$

and C is a positive constant independent of n and $M(g)$.

Throughout the paper, the letter C denotes a positive constant which may differ at each different occurrence, even in the same chain of inequalities. Let δ be the positive constant which is defined by (2.19). We often use the expression (2.20).

Proof of Lemma 3.1. (i) Let $K=\{k; |x-x_{kn}|<\delta a_n/n\}$. Then, the number of K is at most four. By (2.11)

$$W^{-1}(x_{kn})\,|W(x)\,\ell_{kn}(x)|\leqslant C, \qquad x\in R,$$

therefore, using (3.1) and (2.12)

$$\begin{aligned}
\sum^{1}(x)&=(1+|x|)^{-(\Delta+\nu m/6)}\,W^{\nu}(x)\sum_{k\in K}|g(x_{kn})\,h_{kn}(x)|\\
&\leqslant(1+|x|)^{-(\Delta+\nu m/6)}\sum_{k\in K}|W^{-1}(x_{kn})\,W(x)\,\ell_{kn}(x)|^{\nu}\,|W^{\nu}(x_{kn})\,g(x_{kn})|\\
&\quad\times\sum_{j=0}^{\nu-1}(n/a_n)^{j}\,(a_n/n)^{j}\\
&\leqslant C\varepsilon\sum_{k\in K}(1+|x_{kn}|)^{-(\Delta+\alpha+m-\eta+\langle\nu m/6\rangle+\nu m/6)}\qquad(\text{by }|x|\sim|x_{kn}|)\\
&\leqslant C\varepsilon\sum_{k\in K}(1+|x_{kn}|)^{-(\Delta+\langle\!\langle\alpha+\langle\nu m/6\rangle-\nu m/6\rangle\!\rangle+m-\eta+\nu m/3)}\\
&\leqslant C\varepsilon.
\end{aligned}$$

Consequently, we assume that $|x-x_{kn}|\geqslant\delta a_n/n$ below. Using (2.10) (or (2.7)) we rewrite $\sum(x)$ of (3.2) as

$$\begin{aligned}
\sum(x)&=(1+|x|)^{-(\Delta+\nu m/6)}\sum_{k=1}^{n}|W(x)\,p_n(x)/\{W(x_{kn})\,p_n'(x_{kn})\}|^{\nu}\\
&\quad\times|W^{\nu}(x_{kn})\,g(x_{kn})|\sum_{j=0}^{\nu-1}|e_{jk}(x-x_{kn})^{j-\nu}|\\
&\leqslant C\varepsilon(1+|x|)^{-(\Delta+\nu m/6)}\\
&\quad\times\sum_{k=1}^{n}|a_n^{1/2}\,W(x)\,p_n(x)/[na_n^{-1}\{\max(n^{-2/3},1-|x_{kn}|/a_n)\}^{1/4}]|^{\nu}\\
&\quad\times(1+|x_{kn}|)^{-(\alpha+m-\eta+\langle\nu m/6\rangle)}\sum_{j=0}^{\nu-1}|e_{jk}(x-x_{kn})^{j-\nu}|,
\end{aligned}$$

therefore, by (2.12), (2.13), and (2.14)

$$\sum(x) \leqslant C\varepsilon(1+|x|)^{-(\Delta+\nu m/6)} \times \sum_{k=1}^{n} [(x-x_{kn})^{-2}+|x-x_{kn}|^{-1}\{a_n^{-2}|x_{kn}|+|x_{kn}|^{m-1}\}](a_n/n)^2 \times (1+|x_{kn}|)^{-(m-\eta)}(1+|x_{kn}|)^{-(\alpha+\langle \nu m/6\rangle)} \times |a_n^{1/2}W(x)\,p_n(x)/\{\max(n^{-2/3}, 1-|x_{kn}|/a_n)\}^{1/4}|^{\nu}. \tag{3.4}$$

Let $0<\beta<1$. We use (2.5) and (2.6).

(ii) We consider the sum $\sum^2(x)$ for the case of $|x_{kn}| \leqslant \beta a_n$, $|x| \leqslant \beta a_n$. By (3.4),

$$\sum^2(x) \leqslant C\varepsilon \sum_{k \neq n/2}^{2} [t(k,x)^{-2}+|t(k,x)|^{-1}|n/2-k|^{\eta-1}] \times (1+|x_{kn}|)^{-(\alpha+\langle \nu m/6\rangle)}(1+|x|)^{-(\Delta+\nu m/6)}$$
$$(\text{by } 1+|x_{kn}| \geqslant C|n/2-k|(a_n/n))$$
$$\leqslant C\varepsilon.$$

(iii) We consider the sum $\sum^3(x)$ for the case of $|x_{kn}| \geqslant \beta a_n/2$, $|x| \geqslant \beta a_n/2$. Let $|x| \leqslant 2a_n$, then we see that $|x| \sim |x_{kn}| \sim a_n$. By (3.4),

$$\sum^3(x) \leqslant C\varepsilon \sum^3 [t(k,x)^{-2}+|t(k,x)|^{-1}|n/2-k|^{\eta-1}] \times (1+|x_{kn}|)^{-(\Delta+\alpha+\langle \nu m/6\rangle-\nu m/6)}$$
$$\leqslant C\varepsilon a_n^{-(\Delta+\langle\langle \alpha+\langle \nu m/6\rangle-\nu m/6\rangle\rangle)}$$
$$\leqslant C\varepsilon.$$

If $2a_n \leqslant |x|$, then by (2.5) we see that $|a_n^{1/2}W(x)\,p_n(x)| \leqslant C$. Therefore by (3.4),

$$\sum^3(x) \leqslant C\varepsilon \sum^3 [t(k,x)^{-2}+|t(k,x)|^{-1}|n/2-k|^{\eta-1}] \times (1+|x_{kn}|)^{-(\alpha+\langle \nu m/6\rangle-\nu m/6)}(1+|x|)^{-(\Delta+\nu m/6)}$$
$$\leqslant C\varepsilon.$$

(iv) Let $|x_{kn}| \leqslant \beta a_n/2$, $\beta a_n \leqslant |x| \leqslant 2a_n$, and let us denote the sum with respect to these x_{kn} and x by $\sum^4(x)$. By (3.4),

$$\begin{aligned}
\sum^{4}(x) &\leqslant C\varepsilon(1+|x|)^{-\Delta} \sum^{4} [a_n^{-2}+a_n^{-1}](a_n/n)^2 (1+|x_{kn}|)^{-(\alpha+1-\eta+\langle vm/6\rangle)} \\
&\leqslant C\varepsilon(1+|x|)^{-\Delta}(1/n)\int_0^{\beta a_n}(1+t)^{-(\alpha+1-\eta+\langle vm/6\rangle)}\,dt \qquad \text{(by (2.4))} \\
&\leqslant C\varepsilon(1+|x|)^{-\Delta} a_n^{-m} \times \begin{cases} 1, & \alpha-\eta+\langle vm/6\rangle>0, \\ \log(1+n), & \alpha-\eta+\langle vm/6\rangle=0, \\ a_n^{-(\alpha-\eta+\langle vm/6\rangle)}, & \alpha-\eta+\langle vm/6\rangle<0. \end{cases}
\end{aligned}$$

If $\Delta \geqslant 0$, then by $\alpha+\langle vm/6\rangle-\eta+m>0$ we see that

$$\sum^{4}(x) \leqslant C\varepsilon.$$

If $\Delta<0$, then we see that

$$\sum^{4}(x) \leqslant C\varepsilon a_n^{-(\Delta+m)} \times \begin{cases} 1, & \alpha-\eta+\langle vm/6\rangle>0, \\ \log(1+n), & \alpha-\eta+\langle vm/6\rangle=0, \\ a_n^{-(\alpha-\eta+\langle vm/6\rangle)}, & \alpha-\eta+\langle vm/6\rangle<0. \end{cases}$$

Since

$$\Delta+m>\Delta+1\geqslant 0,$$

$$\Delta+m+\alpha-\eta+\langle vm/6\rangle \geqslant \Delta+\langle\!\langle \alpha+\langle vm/6\rangle - vm/6\rangle\!\rangle + vm/6-\eta+m>0,$$

we have

$$\sum^{4}(x) \leqslant C\varepsilon.$$

(v) Let $|x_{kn}| \leqslant \beta a_n/2$, $2a_n \leqslant |x|$, and let us denote the sum with respect to these x_{kn} and x by $\sum^5(x)$. By (3.4)

$$\begin{aligned}
\sum^{5}(x) &\leqslant C\varepsilon(1+|x|)^{-(\Delta+vm/6)} \\
&\quad \times \sum^{5} [a_n^{-2}+a_n^{-1}](a_n/n)^2 (1+|x_{kn}|)^{-(\alpha+1-\eta+\langle vm/6\rangle)} \\
&\leqslant C\varepsilon(1/n)\int_0^{\beta a_n}(1+t)^{-(\alpha+1-\eta+\langle vm/6\rangle)}\,dt \qquad (\text{by } \Delta+vm/6\geqslant 0) \\
&\leqslant C\varepsilon a_n^{-m} \times \begin{cases} 1, & \alpha-\eta+\langle vm/6\rangle>0, \\ \log(1+n), & \alpha-\eta+\langle vm/6\rangle=0, \\ a_n^{-(\alpha-\eta+\langle vm/6\rangle)}, & \alpha-\eta+\langle vm/6\rangle<0. \end{cases}
\end{aligned}$$

Since $\alpha + \langle vm/6 \rangle - \eta + m > 0$ we see that

$$\sum^{5} (x) \leqslant C\varepsilon.$$

(vi) Let $|x| \leqslant \beta a_n/2$, $\beta a_n \leqslant |x_{kn}|$, and let us denote the sum with respect to these x_{kn} and x by $\sum^6 (x)$. By (3.4),

$$\begin{aligned}\sum^{6} (x) &\leqslant C\varepsilon(1+|x|)^{-(\Delta + vm/6)} \sum^{6} [a_n^{-2} + a_n^{-1}](a_n/n)^2 \\ &\quad \times (1+|x_{kn}|)^{-(\alpha+1-\eta+\langle vm/6\rangle - vm/6)} \\ &\leqslant C\varepsilon(1/n) \int_0^{\beta a_n} (1+t)^{-(\alpha+1-\eta+\langle vm/6\rangle - vm/6)}\, dt \qquad (\text{by } \Delta + vm/6 \geqslant 0) \\ &\leqslant C\varepsilon a_n^{-m} \times \begin{cases} 1, & \alpha - \eta + \langle vm/6\rangle - vm/6 > 0, \\ \log(1+n), & \alpha - \eta + \langle vm/6\rangle - vm/6 = 0, \\ a_n^{-(\alpha-\eta+\langle vm/6\rangle - vm/6)}, & \alpha - \eta + \langle vm/6\rangle - vm/6 < 0. \end{cases}\end{aligned}$$

Since $\alpha + \langle vm/6 \rangle - vm/6 - \eta + m > 0$ we see that

$$\sum^{6} (x) \leqslant C\varepsilon. \quad \blacksquare$$

Proof of Lemma 3.2. For odd number v we can use neither (2.13) nor (2.14). However, if we repeat the same method as the proof of Lemma 3.1, then by (2.12) we obtain the upper bound

$$\sum (x) = (1+|x|)^{-(\Delta + vm/6)}\, W^v(x) \sum_{k=1}^{n} |g(x_{kn})\, h^*_{kn}(x)| < CM(g) \log(1+n). \quad \blacksquare$$

Proof of Theorem 4. Let the assumptions of Theorem 4 be satisfies. By (1.9), there exists a polynomial $P_\varepsilon(x)$ such that

$$(1+|x|)^{\alpha+m-\eta+\langle vm/6\rangle}\, W^v(x)\, |f(x) - P_\varepsilon(x)| < \varepsilon, \qquad x \in R \tag{3.5}$$

(cf. [2, p. 180]). By (2.17), for n large enough we have

$$L_n(v-1, v, P_\varepsilon; x) = P_\varepsilon(x), \qquad x \in R.$$

By $h_{0kn}(v; x) = h_{kn}(v; x)$,

$$\begin{aligned}&(1+|x|)^{-(\Delta + vm/6)}\, W^v(x)[L_n(v, f; x) - f(x)] \\ &\quad = (1+|x|)^{-(\Delta + vm/6)}\, W^v(x) \bigg[L_n(v, f - P_\varepsilon; x) + P_\varepsilon(x) - f(x) \\ &\qquad + \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_\varepsilon^{(s)}(x_{kn})\, h_{skn}(x) \bigg].\end{aligned}$$

By Lemma 3.1 and (3.5), it is easy to see

$$(1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x)[\,|L_n(\nu, f-P_\varepsilon; x)|+|P_\varepsilon(x)-f(x)|\,] \leqslant C\varepsilon.$$

Therefore, it is enough to show that

$$\lim_{n\to\infty} \left\| (1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x) \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_\varepsilon^{(s)}(x_{kn})\, h_{skn}(x) \right\|_{C(R)} = 0. \tag{3.6}$$

By (2.16) and (2.18),

$$\begin{aligned}
&(1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x) \left| \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_\varepsilon^{(s)}(x_{kn})\, h_{skn}(x) \right| \\
&\quad \leqslant C(1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x) \sum_{k=1}^{n} \sum_{s=1}^{\ell} |P_\varepsilon^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)| \\
&\qquad \times \sum_{j=s}^{\nu-1} (n/a_n)^{j-s}\, |x-x_{kn}|^{j} \\
&\quad \leqslant C \sum_{s=1}^{\ell} (a_n/n)^{s}\, (1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x) \sum_{k=1}^{n} |P_\varepsilon^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)| \\
&\qquad \times \sum_{j=s}^{\nu-1} (n/a_n)^{j}\, |x-x_{kn}|^{j} \\
&\quad \leqslant C(a_n/n) \sum\nolimits_{7}(x), \qquad \text{say.}
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
\sum\nolimits_{7}(x) = \sum_{s=1}^{\ell} (1+|x|)^{-(\Delta+\nu m/6)}\, W^{\nu}(x) \sum_{k=1}^{n} |P_\varepsilon^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)| \\
\times \sum_{j=0}^{\nu-1} (n/a_n)^{j}\, |x-x_{kn}|^{j}.
\end{aligned}$$

Now, since $P_\varepsilon(x)$ is a polynomial defined by f and ε, we have

$$\begin{aligned}
(1+|x|)^{\alpha+m-\eta+\langle \nu m/6\rangle}\, W^{\nu}(x)\, |P_\varepsilon^{(s)}(x)| \\
< C(s, \varepsilon, f), \qquad x \in R, \quad s = 1, 2, ..., \ell,
\end{aligned}$$

where $C(s, \varepsilon, f)$ is a positive constant independent of n. Therefore, by Lemma 3.2 we have

$$\sum\nolimits_{7}(x) \leqslant C'(s, \varepsilon, f)\log(1+n), \tag{3.8}$$

where $C'(s, \varepsilon, f)$ is independent of n, and may depend on s, ε, and f. Consequently, by (3.7) and (3.8) we obtain (3.6), therefore (1.10) was shown. ▮

4. PROOF OF THEOREM 5

In the rest of the paper we investigate the mean convergence of the Hermite–Fejér interpolation polynomial $L_n(v, f; x)$. Since for the Lagrange case we have Theorem 3, the order v is assumed $v = 2, 3, 4, \ldots$. In this section we obtain a direct theorem, then the following are assumed. Let $1 < p < \infty$, $\alpha > 0$, $\Delta \in R$, and let the conditions (1.11) or (1.12) or (1.13) be satisfied. A real valued continuous function $f \in C(R)$ satisfies (1.14).

LEMMA 4.1 [6, Lemma 2.7]. *Let $0 < \beta < 2$, then, for $x \in R$*

$$W(x) \sum_{|x_{kn}| \geqslant \beta a_n} (1 + |x_{kn}|)^{-\alpha} W^{-1}(x_{kn}) \, |\ell_{kn}(x)| \leqslant C a_n^{-\alpha} \times \begin{cases} 1, & |x| \leqslant \beta a_n/2, \\ |a_n^{1/2} W(x) \, p_n(x)| + \log(1+n), & \beta a_n/2 < |x| \leqslant 2a_n, \\ a_n/|x|, & 2a_n < |x|. \end{cases}$$

Let us define

$$\tilde{h}_{kn}(x) = |\ell_{kn}^{v}(x)| \sum_{j=0}^{v-1} |e_{jk}(x - x_{kn})^j|. \tag{4.1}$$

LEMMA 4.2. *Let $0 < \beta < 2$. Then, for $x \in R$,*

$$\sum(x) = W^v(x) \sum_{|x_{kn}| \geqslant \beta a_n} (1 + |x_{kn}|)^{-\{\alpha + (v-1)m/6\}} W^{-v}(x_{kn}) \, \tilde{h}_{kn}(v; x) \leqslant C a_n^{-\alpha} \times \begin{cases} 1, & |x| \leqslant \beta a_n/2, \\ (|a_n^{1/2} W(x) \, p_n(x)|^{v-1} + 1)\{|a_n^{1/2} W(x) \, p_n(x)| + \log(1+n)\}, & \beta a_n/2 < |x| \leqslant 2a_n, \\ a_n/|x|, & 2a_n < |x|. \end{cases} \tag{4.2}$$

Proof. First, we set

$$\sum(x) = \left\{ W(x) \sum_{|x_{kn}| \geqslant \beta a_n} (1 + |x_{kn}|)^{-\alpha} W^{-1}(x_{kn}) \, |\ell_{kn}(x)| \right\} A_k(x), \tag{4.3}$$

where

$$A_k(x)=|W^{-1}(x_{kn})\ W(x)\ \ell_{kn}(x)|^{\nu-1}\sum_{j=0}^{\nu-1}|e_{jk}(x-x_{kn})^j|\ (1+|x_{kn}|)^{-(\nu-1)\,m/6}.$$

Then, we show that

$$A_k(x)=C\times\begin{cases}1, & |x|\leqslant\beta a_n/2 \quad \text{or} \quad 2a_n<|x|,\\ (|a_n^{1/2}W(x)\ p_n(x)|^{\nu-1}+1), & \beta a_n/2<|x|\leqslant 2a_n.\end{cases} \tag{4.4}$$

We note (2.12). For $|x-x_{kn}|<\delta a_n/n$, we use (2.11).

$$\begin{aligned}A_k(x)&\leqslant C\,|W^{-1}(x_{kn})\ W(x)\ \ell_{kn}(x)|^{\nu-1}\\ &\qquad\times(1+|x_{kn}|)^{-(\nu-1)\,m/6}\sum_{j=0}^{\nu-1}(n/a_n)^j\,|x-x_{kn}|^j\\ &\leqslant C(1+|x_{kn}|)^{-(\nu-1)\,m/6}\sum_{j=0}^{\nu-1}(n/a_n)^j\,(a_n/n)^j\leqslant C.\end{aligned} \tag{4.5}$$

Let $|x|\leqslant\beta a_n/2$ or $2a_n<|x|$, and $|x-x_{kn}|\geqslant\delta a_n/n$. Then, by (2.5) and (2.7),

$$\begin{aligned}A_k(x)&\leqslant C\,|a_n^{1/2}W(x)\ p_n(x)/[(x-x_{kn})\,na_n^{-1}\\ &\qquad\times\{\max(n^{-2/3},1-|x_{kn}|/a_n)\}^{1/4}]|^{\nu-1}\,(1+|x_{kn}|)^{-(\nu-1)\,m/6}\\ &\qquad\times\sum_{j=0}^{\nu-1}(n/a_n)^j\,|x-x_{kn}|^j\\ &\leqslant C\{\max(n^{-2/3},1-|x_{kn}|/a_n)\}^{-(\nu-1)/4}\,(1+|x_{kn}|)^{-(\nu-1)\,m/6}\\ &\leqslant C.\end{aligned} \tag{4.6}$$

If $\beta a_n/2<|x|\leqslant 2a_n$ and $|x-x_{kn}|\geqslant\delta a_n/n$, then we have

$$\begin{aligned}A_k(x)&\leqslant C\,|a_n^{1/2}W(x)\ p_n(x)/[(x-x_{kn})\,na_n^{-1}\\ &\qquad\times\{\max(n^{-2/3},1-|x_{kn}|/a_n)\}^{1/4}]|^{\nu-1}\\ &\qquad\times(1+|x_{kn}|)^{-(\nu-1)\,m/6}\sum_{j=0}^{\nu-1}(n/a_n)^j\,|x-x_{kn}|^j\\ &\leqslant C\,|a_n^{1/2}W(x)\ p_n(x)|^{\nu-1}\{\max(n^{-2/3},1-|x_{kn}|/a_n)\}^{-(\nu-1)/4}\\ &\qquad\times(1+|x_{kn}|)^{-(\nu-1)\,m/6}\\ &\leqslant C\,|a_n^{1/2}W(x)\ p_n(x)|^{\nu-1}.\end{aligned} \tag{4.7}$$

Therefore, by (4.5), (4.6), and (4.7) we obtain (4.4), consequently (4.3).

Applying Lemma 4.1 to (4.3) we obtain (4.2). ∎

LEMMA 4.3 (cf. [6, Lemma 3.1]). *We set* $0<\beta<2$, *and we let* $n=1, 2, 3, \ldots$. *If* $f_n(x)=0$ *for* $|x|<\beta a_n$, *furthermore,*

$$|W^{\nu}(x)\, f_n(x)| \leqslant \varepsilon(1+|x|)^{-\{\alpha+(\nu-1)\, m/6\}}, \qquad x\in R,$$

then we have

$$\limsup_{n\to\infty} \|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, f_n; x)\|_{L_p(R)} \leqslant C\varepsilon. \tag{4.8}$$

Proof. By Lemma 4.2

$$\begin{aligned}
&|W^{\nu}(x)\, L_n(\nu, f_n; x)| \\
&\quad\leqslant \varepsilon W^{\nu}(x) \sum_{|x_k|\geqslant \beta a_n} (1+|x_{kn}|)^{-\{\alpha+(\nu-1)\, m/6\}}\, W^{-\nu}(x_{kn})\, \tilde{h}_{kn}(\nu; x) \\
&\quad\leqslant C\varepsilon a_n^{-\alpha} \times \begin{cases} 1, & |x|\leqslant \beta a_n/2, \\ (|a_n^{1/2} W(x)\, p_n(x)|^{\nu-1}+1)\{|a_n^{1/2} W(x)\, p_n(x)| + \log(1+n)\}, & \\ & \beta a_n/2<|x|\leqslant 2a_n, \\ a_n/|x|, & 2a_n<|x|. \end{cases}
\end{aligned} \tag{4.9}$$

We repeat the same method as the proof of [6, Lemma 3.1] below. From (4.9),

$$\begin{aligned}
\tau_n^{(1)} &= \|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, f_n; x)\|_{L_p(|x|\leqslant \beta a_n/2)} \\
&\leqslant C\varepsilon a_n^{-\alpha}\, \|(1+|x|^{-\Delta}\|_{L_p(|x|\leqslant \beta a_n/2)} \\
&\leqslant C\varepsilon a_n^{-\alpha} \times \begin{cases} 1, & \Delta p>1, \\ \{\log(1+n)\}^{1/p}, & \Delta p=1 \\ a_n^{1/p-\Delta}, & \Delta p<1. \end{cases}
\end{aligned}$$

Here, we see that all conditions of (1.11), (1.12), and (1.13) imply

$$1/p-(\alpha+\Delta)\leqslant 1/p-(\langle\!\langle \alpha \rangle\!\rangle+\Delta)<0. \tag{4.10}$$

Therefore,

$$\tau_n^{(1)} \leqslant C\varepsilon.$$

Next, we estimate

$$\tau_n^{(2)} = \|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, f_n; x)\|_{L_p(\beta a_n/2\leqslant |x|\leqslant 2a_n)}.$$

Using Lemma 4.2, we have, again

$$\begin{aligned}\tau_n^{(2)} \leqslant C\varepsilon a_n^{-\alpha}[\, & a_n^{\nu/2-\Delta}\, \| W(x)\, p_n(x) \|^{\nu}_{L_{p\nu}(\beta a_n/2 \leqslant |x| \leqslant 2a_n)} \\ & + a_n^{1/2-\Delta}\, \| W(x)\, p_n(x) \|_{L_p(\beta a_n/2 \leqslant |x| \leqslant 2a_n)} \\ & + \{\log(1+n)\}\, a_n^{(\nu-1)/2-\Delta}\, \| W(x)\, p_n(x) \|^{\nu-1}_{L_{p(\nu-1)}(\beta a_n/2 \leqslant |x| \leqslant 2a_n)} \\ & + \{\log(1+n)\}\, a_n^{1/p-\Delta}].\end{aligned}$$

Since, by (2.2) and (2.9),

$$\| W(x)\, p_n(x) \|_{L_p(R)} \sim a_n^{1/p-1/2} \times \begin{cases} 1, & p<4, \\ \{\log(1+n)\}^{1/4}, & p=4, \\ a_n^{(m/6)(1-4/p)}, & p>4, \end{cases}$$

we have

$$\begin{aligned}\tau_n^{(2)} \leqslant C\varepsilon a_n^{1/p-(\alpha+\Delta)} \times \Bigg[& \begin{cases} 1, & 1<p<4/\nu, \\ \{\log(1+n)\}^{\nu/4}, & p=4/\nu, \\ a_n^{(m/6)(\nu-4/p)}, & p>4/\nu, \end{cases} \\ & + \begin{Bmatrix} 1, & 1<p<4, \\ \{\log(1+n)\}^{1/4}, & p=4, \\ a_n^{(m/6)(1-4/p)}, & p>4, \end{Bmatrix} \\ & + \{\log(1+n)\} \times \begin{Bmatrix} 1, & 1<p<4/(\nu-1) \\ \{\log(1+n)\}^{(\nu-1)/4}, & p=4/(\nu-1), \\ a_n^{(m/6)(\nu-1-4/p)}, & p>4/(\nu-1), \end{Bmatrix} \\ & + \{\log(1+n)\} \Bigg].\end{aligned}$$

Therefore, by our assumption (1.11) or (1.12), or (1.13),

$$\tau_n^{(2)} \leqslant C\varepsilon.$$

Finally, from (4.2),

$$\begin{aligned}\tau_n^{(3)} &= \| (1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, f_n; x) \|_{L_p(|x| \geqslant 2a_n)} \\ &\leqslant C\varepsilon a_n^{-\alpha+1}\, \| |x|^{-1}\, (1+|x|)^{-\Delta} \|_{L_p(|x| \geqslant 2a_n)}.\end{aligned}$$

Therefore, by (4.10),

$$\tau_n^{(3)} \leqslant C\varepsilon a_n^{1/p-(\alpha+\Delta)} \leqslant C\varepsilon.$$

Consequently, we obtained (4.8), that is, the proof of Lemma 4.3 is complete. ∎

LEMMA 4.4 (cf. [6, Lemma 3.2]). *Let* $\varepsilon > 0$, $0 < \beta < 1$. *We assume that* $\Psi_n \in C(R)$, $n = 1, 2, 3, ...,$ *are the functions satisfying*

$$\Psi_n(x) = 0, \qquad |x| \geqslant \beta a_n,$$

and

$$|W^\nu(x)\, \Psi_n(x)| \leqslant \varepsilon(1+|x|)^{-\{\alpha+(\nu-1)\,m/6\}}, \qquad x \in R.$$

Then,

$$\limsup_{n\to\infty} \|(1+|x|)^{-\Delta}\, W^\nu(x)\, L_n(\nu, \Psi_n; x)\|_{L_p(|x| \geqslant 2\beta a_n)} \leqslant C\varepsilon,$$

where C *is independent of* ε, n, *and* Ψ_n.

Proof. We see that

$$\begin{aligned}&|W^\nu(x)\, L_n(\nu, \Psi_n; x)|\\ &\quad\leqslant \varepsilon \sum_{|x_{kn}| \leqslant \beta a_n} |W^{-1}(x_{kn})\, W(x)\, \ell_{kn}(x)|\, (1+|x_{kn}|)^{-\alpha}\, A_k(x),\end{aligned}$$

where $A_k(x)$ is given by (4.3). Then, by (4.5), (4.6), and (4.7),

$$A_k(x) \leqslant C(|a_n^{1/2}\, W(x)\, p_n(x)|^{\nu-1}+1).$$

Since $|x| \geqslant 2\beta a_n$ and $|x_{kn}| \leqslant \beta a_n$, we obtain $|x_{kn}-x| \sim |x|$. Hence, by (2.10),

$$\begin{aligned}&|W^\nu(x)\, L_n(\nu, \Psi_n; x)|\\ &\quad\leqslant C\varepsilon(|a_n^{1/2}\, W(x)\, p_n(x)|^{\nu-1}+1)\\ &\qquad\times \sum_{|x_{kn}| \leqslant \beta a_n} |W^{-1}(x_{kn})\, W(x)\, \ell_{kn}(x)|\, (1+|x_{kn}|)^{-\alpha}\\ &\quad\leqslant C\varepsilon(|a_n^{1/2}\, W(x)\, p_n(x)|^{\nu}+|a_n^{1/2}\, W(x)\, p_n(x)|)\, |x|^{-1}\\ &\qquad\times (a_n/n) \sum_{|x_{kn}| \leqslant \beta a_n} (1+|x_{kn}|)^{-\alpha}\end{aligned}$$

$$\begin{aligned}
&\leqslant C\varepsilon(|a_n^{1/2} W(x)\, p_n(x)|^{\nu} + |a_n^{1/2} W(x)\, p_n(x)|)\, |x|^{-1} \\
&\quad \times \sum_{|x_{kn}| \leqslant \beta a_n} (1+|x_{kn}|)^{-\alpha}\, (x_{k-1,n} - x_{k+1,n}) \qquad \text{(by (2.4))} \\
&\leqslant C\varepsilon(|a_n^{1/2} W(x)\, p_n(x)|^{\nu} + |a_n^{1/2} W(x)\, p_n(x)|)\, |x|^{-1} \\
&\quad \times \int_{-2\beta a_n}^{2\beta a_n} (1+|t|)^{-\alpha}\, dt \\
&\leqslant C\varepsilon(|a_n^{1/2} W(x)\, p_n(x)|^{\nu} + |a_n^{1/2} W(x)\, p_n(x)|)\, |x|^{-1}\, a_n^{1-\langle\!\langle \alpha \rangle\!\rangle} (\log n)^*,
\end{aligned}$$

where

$$(\log n)^* = \begin{cases} \log(1+n), & \alpha = 1, \\ 1, & \text{otherwise}. \end{cases}$$

Therefore, by (2.9),

$$\begin{aligned}
&\|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, \Psi_n; x)\|_{L_p(|x| \geqslant 2\beta a_n)} \\
&\quad \leqslant C\varepsilon a_n^{1-\langle\!\langle \alpha \rangle\!\rangle} (\log n)^*\, a_n^{-(\Delta+1)} \\
&\qquad \times (\|a_n^{1/2} W(x)\, p_n(x)\|^{\nu}_{L_{p\nu}(R)} + \|a_n^{1/2} W(x)\, p_n(x)\|_{L_p(R)}) \quad \text{(by } \Delta + 1 > 0) \\
&\quad \leqslant C\varepsilon a_n^{1/p - (\Delta + \langle\!\langle \alpha \rangle\!\rangle)} (\log n)^* \\
&\qquad \times \left[\begin{cases} 1, & 1 < p < 4/\nu, \\ \{\log(1+n)\}^{\nu/4}, & p = 4/\nu, \\ n^{(1/6)(\nu - 4/p)}, & p > 4/\nu, \end{cases} \right. \\
&\qquad \left. + \begin{cases} 1, & 1 < p < 4, \\ \{\log(1+n)\}^{1/4}, & p = 4, \\ n^{(1/6)(1-4/p)}, & p > 4, \end{cases} \right] \\
&\quad \leqslant C\varepsilon a_n^{1/p - (\Delta + \langle\!\langle \alpha \rangle\!\rangle)} (\log n)^* \\
&\qquad \times \left[\begin{cases} 1, & 1 < p < 4/\nu, \\ \{\log(1+n)\}^{\nu/4}, & p = 4/\nu, \\ a_n^{(m/6)(\nu - 4/p)}, & p > 4/\nu, \end{cases} \right. \\
&\qquad \left. + \begin{cases} 1, & 1 < p < 4, \\ \{\log(1+n)\}^{1/4}, & p = 4, \\ a_n^{(m/6)(1-4p)}, & p > 4, \end{cases} \right] \\
&\quad \leqslant C\varepsilon \qquad \text{(by (1.11) or (1.12) or (1.13)).} \quad \blacksquare
\end{aligned}$$

LEMMA 4.5 (cf. [6, Lemma 3.4]). *Let* $\varepsilon > 0$, $0 < \beta < 1/2$, *and assume that* $\Psi_n(x) \in C(R)$, $n = 1, 2, 3, \ldots$, *are the functions satisfying*

$$\Psi_n(x) = 0, \qquad |x| \geqslant \beta a_n,$$

and

$$|W^{\nu}(x)\, \Psi_n(x)| < \varepsilon(1+|x|)^{-\{\alpha+(\nu-1)\,m/6\}}, \qquad x \in R, \quad n \geqslant 1.$$

Then,

$$\limsup_{n\to\infty} \|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, \Psi_n; x)\|_{L_p(|x| \leqslant 2\beta a_n)} \leqslant C\varepsilon.$$

Proof. By definition

$$\begin{aligned}
&|W^{\nu}(x)\, L_n(\nu, \Psi_n; x)| \\
&\quad\leqslant \varepsilon \sum_{|x_{kn}| \leqslant \beta a_n} |(1+|x_{kn}|)^{-\alpha}\, W^{-1}(x_{kn})\, W(x)\, \ell_{kn}(x)|\, A_k(x) \\
&\quad\leqslant C\varepsilon \sum_{|x_{kn}| \leqslant \beta a_n} |(1+|x_{kn}|)^{-\alpha}\, W^{-1}(x_{kn})\, W(x)\, \ell_{kn}(x)|,
\end{aligned}$$

where $A_k(x)$ is defined by (4.3), and then, $A_k(x) \leqslant C$, $x \leqslant 2\beta a_n$. We use the expression (2.20). By (2.7), (2.11), and (2.5),

$$\begin{aligned}
&|W^{\nu}(x)\, L_n(\nu, \Psi_n; x)| \\
&\quad\leqslant C\varepsilon \sum_{|x_{kn}| \leqslant \beta a_n,\ t(k,x) \neq 0} (1+|x_{kn}|)^{-\alpha}\, |a_n^{1/2}\, W(x)\, p_n(x)/t(k,x)| \\
&\quad\leqslant C\varepsilon \sum_{|x_{kn}| \leqslant \beta a_n,\ t(k,x) \neq 0} (1+|x_{kn}|)^{-\alpha}\, |1/t(k,x)|.
\end{aligned}$$

Therefore, we have

$$|W^{\nu}(x)\, L_n(\nu, \Psi_n; x)| \leqslant C\varepsilon\{\log(1+n)\}. \tag{4.11}$$

By (4.11),

$$\begin{aligned}
&\|(1+|x|)^{-\Delta}\, W^{\nu}(x)\, L_n(\nu, \Psi_n; x)\|_{L_p(|x| \leqslant 2\beta a_n)} \\
&\quad\leqslant C\varepsilon\{\log(1+n)\}\, \|(1+|x|)^{-\Delta}\|_{L_p(|x| \leqslant 2\beta a_n)} \\
&\quad\leqslant C\varepsilon\{\log(1+n)\}\, a_n^{1/p-\Delta} \qquad \text{(by (1.11), (1.12), and (1.13))} \\
&\quad\leqslant C\varepsilon.
\end{aligned}$$

Consequently, we see that the proof of Lemma 4.5 is complete. ∎

Remark 4.6. In the above consideration of Section 4 we can replace $\tilde{h}_{kn}(x)$ in (4.1) by $h^*_{kn}(x) = |\ell^{\nu}_{kn}(x)| \sum_{j=0}^{\nu-1} (n/a_n)^j |x - x_{kn}|^j$ (defined in (3.3)).

Proof of Theorem 5. By (1.14) there exists a polynomial $P_\varepsilon(x)$ such that

$$|(1+|x|)^{\alpha+(\nu-1)\,m/6}\, W^\nu(x)\{f(x) - P_\varepsilon(x)\}| < \varepsilon, \qquad x \in R$$

(cf. [2. p. 180]). Since (by (2.17)),

$$L_n(\nu-1, \nu, P_\varepsilon; x) = P_\varepsilon(x) \qquad \text{and} \qquad h_{0kn}(\nu; x) = h_{kn}(\nu; x), \qquad x \in R,$$

we have

$$\begin{aligned}
&(1+|x|)^{-\Delta}\, W^\nu(x)[L_n(\nu, f; x) - f(x)] \\
&\quad = (1+|x|)^{-\Delta}\, W^\nu(x)\Bigg[L_n(\nu, f - P_\varepsilon; x) + \{P_\varepsilon(x) - f(x)\} \\
&\qquad + \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_\varepsilon^{(s)}(x_{kn})\, h_{skn}(x)\Bigg] \\
&\quad = \sum_1 (x) + \sum_2 (x) + \sum_3 (x).
\end{aligned}$$

Let $\chi[-a_n/4, a_n/4]$ denote the characteristic function of $[-a_n/4, a_n/4]$ and write

$$\begin{aligned}
f - p_\varepsilon &= (f - p_\varepsilon)\,\chi[-a_n/4, a_n/4] + (f - p_\varepsilon)(1 - \chi[-a_n/4, a_n/4]) \\
&= \Psi_n + f_n.
\end{aligned}$$

Applying Lemma 4.3, 4.4, and 4.5 to f_n or Ψ_n, we obtain

$$\Big\|\sum_1 (x)\Big\|_{L_p(R)} \leqslant C\varepsilon.$$

Since, by (4.10) we see that $-p\{\Delta + \alpha + (\nu-1)\,m/6\} < -p(\Delta + \alpha) < -1$, we also have

$$\Big\|\sum_2 (x)\Big\|_{L_p(R)} \leqslant C\varepsilon\, \|(1+|x|)^{-\{\Delta+\alpha+(\nu-1)\,m/6\}}\|_{L_p(R)} \leqslant C\varepsilon.$$

Finally, we estimate $\sum_3 (x)$. We see that

$$\left|(1+|x|)^{-\Delta}\, W^{\nu}(x) \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_{\varepsilon}^{(s)}(x_{kn})\, h_{skn}(x)\right|$$

$$\leqslant C \sum_{s=1}^{\ell} (1+|x|)^{-\Delta}\, W^{\nu}(x) \sum_{k=1}^{n} |P_{\varepsilon}^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)|$$
$$\times \sum_{j=s}^{\nu-1} (n/a_n)^{j-s}\, |x-x_{kn}|^{j}$$
$$\leqslant C \sum_{s=1}^{\ell} (a_n/n)^{s}\, (1+|x|)^{-\Delta}\, W^{\nu}(x) \sum_{k=1}^{n} |P_{\varepsilon}^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)|$$
$$\times \sum_{j=0}^{\nu-1} (n/a_n)^{j}\, |x-x_{kn}|^{j}$$
$$= C(a_n/n) \sum_3{}' (x),$$

where

$$\sum_3{}' (x) = \sum_{s=1}^{\ell} (1+|x|)^{-\Delta}\, W^{\nu}(x) \sum_{k=1}^{n} |P_{\varepsilon}^{(s)}(x_{kn})\, \ell_{kn}^{\nu}(x)|$$
$$\times \sum_{j=0}^{\nu-1} (n/a_n)^{j}\, |x-x_{kn}|^{j}.$$

Here, $P_{\varepsilon}(x)$ is defined by only ε and f, therefore there exists a positive constant $M(s, \varepsilon, f)$ such that

$$|W^{\nu}(x)\, P_{\varepsilon}^{(s)}(x)| \leqslant M(s, \varepsilon, f)(1+|x|)^{-\{\alpha+(\nu-1)\, m/6\}}, \qquad s=1, 2, ..., \ell.$$

Let $0<\beta<1$, and let us define

$$f_{s\varepsilon n}(x) = p_{\varepsilon}^{(s)}(x)(1-\chi[-\beta a_n, \beta a_n])$$

and

$$\Psi_{s\varepsilon n}(x) = p_{\varepsilon}^{(s)}(x)\, \chi[-\beta a_n, \beta a_n]$$

for each $s=1, 2, ..., \ell$. Since by Remark 4.6 we can apply Lemma 4.3, Lemma 4.4, and lemma 4.5 to $f_{s\varepsilon n}$ or $\Psi_{s\varepsilon n}$, we have

$$\left\| \sum_3{}' (x) \right\|_{L_p(R)} \leqslant C \sum_{s=1}^{\ell} M(s, \varepsilon, f)$$

(we replace ε to $M(s, \varepsilon, f)$ in each lemma). Consequently, we see that the proof of Theorem 5 is complete. ∎

5. PROOF OF THEOREM 6

In this section we let $v = 3, 5, 7, \ldots$, and we will obtain an inverse theorem. We need the following lemmas.

LEMMA 5.1 [6, Lemma 2.5]. *Let $\xi \in R$. There exists $C > 0$ such that for $\lambda \geqslant 2$, there exist polynomials P_λ^* of degree $\leqslant C\lambda \log \lambda$ satisfying*

$$P_\lambda^*(t) \sim (1+t^2)^\xi,$$

uniformly for $-\lambda \leqslant t \leqslant \lambda$.

LEMMA 5.2 [6, Lemma 3.5]. *Let $0 < \sigma < 1$, $0 < \theta < 1 - \sigma$, and $1 < p < \infty$. Then, there exists C such that for $n \geqslant 1$ and P of degree at most θn, we have*

$$\|P\|_{L_p[-a_{\sigma n}, a_{\sigma n}]} \leqslant C a_n^{1/2} \sum_{j=n-1}^{n} \|p_j W P\|_{L_p[-a_n, a_n]}.$$

The following proposition is important itself, and to prove Theorem 6 we use it as one of the lemmas. We use the number $q_n = (2n/m)^{1/m}$ instead of a_n, defined in Section 2 (see (2.2)). Let $\beta = (1/2)\{\pi^{1/2}\Gamma(m/2)/\Gamma(m+1/2)\}^{1/m}$ be Freud's constant, and let $\alpha = m(m/2)^{(m-1)/m} \binom{m-2}{m/2-1} \beta^{m-1}$.

In [3], we showed that the proposition held for $x_{kn} \in [\theta, \Theta]$, where θ and Θ are positive constants. We omit the proof of Proposition 5.3, because we can show it by careful repeating the same line of the consideration as one in [3].

PROPOSITION 5.3 (cf. [3, Lemma 14]). *For $j = 0, 1, 2, \ldots$, there exists a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j \Psi_j(-v) > 0$ for $v = 1, 2, 3, \ldots$, and the following relation holds: Let $0 < \varepsilon < 1$. Then, we have an expression*

$$e_{2s,k} = (-1)^s \{1/(2s)!\} \Psi_s(-v) \alpha^{2s} q_n^{2s(m-1)} \{1 + \eta_{kn}(v, s\}, \tag{5.1}$$

where $\eta_{kn}(v, s)$ satisfies

$$|\eta_{kn}(v, s)| \leqslant C\varepsilon^2, \tag{5.2}$$

for k with $|x_{kn}| \leqslant \varepsilon q_n$ and $s = 0, 1, \ldots, \tilde{v}$. Here, the positive constant C is independent of n, k, and ε, and may depend on v, s, and m; $\tilde{v}$ is the largest integer not exceeding $(v-1)/2$.

Proof of Theorem 6. Let $\nu = 3, 5, 7, \ldots$. We repeat the line of [6, proof of the necessary conditions of Theorem 1.3]. Let $\zeta(x)$ be an even continuous function that is decreasing in $[0, \infty)$, with

$$\zeta(x) \geqslant \{\log(2 + |x|)\}^{-1/(2p)} \qquad (x \in R), \qquad \lim_{x \to \infty} \zeta(x) = 0.$$

Let us define two spaces: X consists of all continuous functions satisfying

$$\|f\|_X = \|(1 + |x|)^{\alpha + (\nu - 1) m/6} W^\nu(x) f(x) \zeta^{-1}(x)\|_{C(R)} < \infty,$$

and Y consists of all measurable functions satisfying

$$\|f\|_Y = \|(1 + |x|)^{-\Delta} W^\nu(x) f(x)\|_{L_p(R)} < \infty.$$

For each $f \in X$, (1.14) is satisfies, so our hypothesis ensures that

$$\lim_{n \to \infty} \|L_n(\nu, f) - f\|_Y = 0.$$

Since X is a Banach space, by the uniform boundedness principle, there exists $C > 0$ such that for $n = 1, 2, 3, \ldots$, and every $f \in X$,

$$\|L_n(\nu, f) - f\|_Y \leqslant C \|f\|_X.$$

Noting $L_1(\nu, f; x) = f(0)$, $x \in R$, we have for every $f \in C(R)$ with $f(0) = 0$ that

$$\|f\|_Y \leqslant C \|f\|_X,$$

consequently, we obtain

$$\|L_n(\nu, f)\|_Y \leqslant C \|f\|_X, \tag{5.3}$$

that is,

$$\|(1 + |x|)^{-\Delta} W^\nu(x) L_n(\nu, f; x)\|_{L_p(R)} \leqslant C \|\zeta^{-1}(x)(1 + |x|)^{\alpha + (\nu - 1) m/6} W^\nu(x) f(x)\|_{C(R)}. \tag{5.4}$$

Let $0 < \varepsilon$ be small enough, and let us consider the function $g_n \in C(R)$ such that $g_n(x) = 0$ in $[0, \infty) \cup (-\infty, -\varepsilon a_n)$;

$$\|g_n\|_X = \|\zeta^{-1}(x)(1 + |x|)^{\alpha + (\nu - 1) m/6} W^\nu(x) g_n(x)\|_{C(R)} = 1; \tag{5.5}$$

and for $-\varepsilon a_n \leqslant x_{kn} < 0$,

$$\zeta^{-1}(x_{kn})(1+|x_{kn}|)^{\alpha+(\nu-1)m/6}\, W^{\nu}(x_{kn})\; g_n(x_{kn})\; \mathrm{sign}\{p_n'(x_{kn})\} = 1.$$

Then, for $x \geqslant 1$, we have

$$|L_n(\nu, g_n; x)| = \Bigg| \sum_{x_{kn} \in [-\varepsilon a_n, 0)} g_n(x_{kn})[p_n(x)/\{(x-x_{kn})\, p_n'(x_{kn})\}]^{\nu} \times \sum_{j=0}^{\nu-1} e_{jk}(x-x_{kn})^j \Bigg|. \tag{5.6}$$

Here, we show that for $\nu \geqslant 3$ and so n large enough,

$$(-1)^{(\nu-1)/2}\{1/(x-x_{kn})^{\nu-1}\sum_{j=0}^{\nu-1} e_{jk}(x-x_{kn})^j \geqslant C(n/a_n)^{\nu-1}. \tag{5.7}$$

In fact, using the expression (2.20) we see that for $x \geqslant 1$ and $x_{kn} \in [-\varepsilon a_n, 0)$,

$$|t(k, x)|\; \delta a_n/n \geqslant x = t(x)\; \delta a_n/n \geqslant 1,$$

where $t(x)$ is a positive number. Therefore, we have

$$|t(k, x)| \geqslant t(x) \geqslant (1/\delta)(n/a_n). \tag{5.8}$$

By (5.1) and (5.2), there exists a positive constant $C(\nu)$ such that

$$(-1)^{(\nu-1)/2}\, e_{\nu-1,k} \geqslant C(\nu)(n/a_n)^{\nu-1}. \tag{5.9}$$

From (5.8) and (5.9),

$$\begin{aligned}
&(-1)^{(\nu-1)/2}\,\{1/(x-x_{kn})\}^{\nu-1}\sum_{j=0}^{\nu-1} e_{jk}(x-x_{kn})^j \\
&\quad = (-1)^{(\nu-1)/2}\left\{e_{\nu-1,k} + \sum_{j=0}^{\nu-2} e_{jk}(x-x_{kn})^{j-\nu+1}\right\} \\
&\quad \geqslant C(\nu)(n/a_n)^{\nu-1} - C\sum_{j=0}^{\nu-2}(n/a_n)^j\,\{|t(k,x)|\,\delta\}^{j-\nu+1}\,(n/a_n)^{\nu-1-j} \\
&\quad = (n/a_n)^{\nu-1}\left[C(\nu) - C\sum_{j=0}^{\nu-2}\{|t(k,x)|\,\delta\}^{j-\nu+1}\right] \\
&\quad \geqslant (n/a_n)^{\nu-1}\,[C(\nu) - C(a_n/n)] \\
&\quad \geqslant C(n/a_n)^{\nu-1}.
\end{aligned}$$

Therefore, we obtain (5.7).

Let $1 \leqslant x \leqslant 2a_n$. Applying (5.7) to (5.6), we have

$$
\begin{aligned}
&|L_n(v, g_n; x)| \\
&\quad \geqslant C \left| \sum_{x_{kn} \in [-\varepsilon a_n, 0)} g_n(x_{kn})[p_n(x)/p_n'(x_{kn})]^v (x - x_{kn})^{-1} (n/a_n)^{v-1} \right| \\
&\quad \geqslant C(a_n/n)\, |a_n^{1/2} p_n(x)|^v \sum_{x_{kn} \in [-\varepsilon a_n, 0)} (1 + |x_{kn}|)^{-\{\alpha + (v-1)\, m/6\}} \\
&\qquad \times \zeta(x_{kn})(x - x_{kn})^{-1} \\
&\quad \geqslant C\zeta(a_n)\, |a_n^{1/2} p_n(x)|^v \sum_{x_{kn} \in [-\varepsilon a_n, 0)} (1 + |x_{kn}|)^{-\{\alpha + (v-1)\, m/6\}} \\
&\qquad \times (x - x_{kn})^{-1} (x_{k-1, n} - x_{k+1, n}) \qquad \text{(by (2.4))} \\
&\quad \geqslant C\zeta(a_n)\, |a_n^{1/2} p_n(x)|^v \int_0^{\varepsilon a_n/2} [(1+t)^{-\{\alpha + (v-1)\, m/6\}}/(x+t)]\, dt \\
&\quad \geqslant C\zeta(a_n)(|a_n^{1/2} p_n(x)|^v/x) \int_0^{\varepsilon a_n/2} (1+t)^{-\{\alpha + (v-1)\, m/6\}}\, dt \\
&\quad \geqslant C\zeta(a_n)(|a_n^{1/2} p_n(x)|^v/x) \\
&\qquad \times \begin{cases} 1, & \alpha + (v-1)\, m/6 > 1, \\ \log(1 + \min(\varepsilon a_n/2, x)), & \alpha + (v-1)\, m/6 = 1, \\ (\min(\varepsilon a_n/2, x))^{1 - \{\alpha + (v-1)\, m/6\}}, & \alpha + (v-1)\, m/6 < 1, \end{cases} \\
&\quad \geqslant C\zeta(a_n)\, |a_n^{1/2} p_n(x)|^v\, x^{-\langle\!\langle \alpha + (v-1)\, m/6 \rangle\!\rangle} (\log x)^{\#},
\end{aligned} \tag{5.10}
$$

where

$$
(\log x)^{\#} = \begin{cases} \log(1+x), & \alpha + (v-1)\, m/6 = 1, \\ 1, & \text{otherwise.} \end{cases}
$$

The last inequality is obtained by considering $1 \leqslant x \leqslant \varepsilon a_n/2$ and $\varepsilon a_n/2 < x \leqslant 2a_n$ separately. Since by (5.3) we see that

$$
\|L_n(v, g_n)\|_Y \leqslant C\, \|g_n\|_X \leqslant C,
$$

we have

$$
\begin{aligned}
C &\geqslant \|(1 + |x|)^{-\Delta}\, W^v(x)\, L_n(v, g_n; x)\|_{L_p(1, 2a_n)} \\
&\geqslant C\{\log(1+n)\}^{-\{1/(2p)\}}\, \|(1 + |x|)^{-(\Delta + \langle\!\langle \alpha + (v-1)\, m/6 \rangle\!\rangle)} \\
&\quad \times |a_n^{1/2}\, W(x)\, p_n(x)|^v\|_{L_p(1, 2a_n)} \qquad \text{(see the definition } \zeta(x)).
\end{aligned} \tag{5.11}
$$

Since by (2.5) we have

$$\|(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)}\,|a_n^{1/2}\,W(x)\,p_n(x)|^\nu\|_{L_p[0,1]}\leqslant C,$$

(5.11) implies that

$$C\geqslant\{\log(1+n)\}^{-\{1/(2p)\}}\,[\,\|(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)}\\ \times|a_n^{1/2}\,W(x)\,p_n(x)|^\nu\|_{L_p(-2a_n,\,2a_n)}-C].$$

Therefore,

$$C\{\log(1+n)\}^{1/(2p)}\geqslant a_n^{\nu/2}\,\|(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)}\\ \times|W(x)\,p_n(x)|^\nu\|_{L_p(-2a_n,\,2a_n)}-C,$$

that is,

$$C\{\log(1+n)\}^{1/(2p)}\geqslant a_n^{\nu/2}\,\|(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)/\nu}\\ \times|W(x)\,p_n(x)|\|^\nu_{L_{p\nu}(-2a_n,\,2a_n)}-C. \tag{5.12}$$

Now, let $P^*_{2a_n}$ be the polynomial of Lemma 5.1 of degree $0(a_n\log a_n)=o(n)$ such that for $|x|\leqslant 2a_n$,

$$P^*_{2a_n}(x)\sim(1+x^2)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)/(2\nu)}\\ \sim(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)/\nu}.$$

We obtain from (5.12) that

$$C\{\log(1+n)\}^{1/(2p\nu)}\geqslant a_n^{1/2}\sum_{j=n-1}^{n}\|W(x)\,p_j(x)\,P^*_{2a_n}(x)\|_{L_{p\nu}(-2a_j,\,2a_j)}-C.$$

In Lemma 5.2 setting $\sigma=1/2$ and $\theta=1/4$, we have

$$C\{\log(1+n)\}^{1/(2p\nu)}\geqslant C\,\|P^*_{2a_n}(x)\|_{L_{p\nu}(-a_{n/2},\,a_{n/2})}-C\\ \geqslant C\,\|(1+|x|)^{-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)/\nu}\|_{L_{p\nu}(-a_{n/2},\,a_{n/2})}-C\\ \geqslant C\times\begin{cases} a_n^{(1/\nu)\{(1/p)-(\Delta+\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle)\}}-C, & \\ \qquad \Delta<(1/p)-\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle, & \\ \{\log(1+n)\}^{1/(p\nu)}-C, & \\ \qquad \Delta=(1/p)-\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle, & \\ 1-C, \qquad \Delta>(1/p)-\langle\!\langle \alpha+(\nu-1)\,m/6\rangle\!\rangle. & \end{cases}$$

However, for these inequalities can occur only the last one, that is, $\Delta > (1/p) - \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle$. Therefore, we obtain the necessary conditions for $1 < p \leqslant 4/\nu$ (but $\nu < 4$).

Next, we consider the case of $p > 4/\nu$. We return to (5.10), that is,

$$|L_n(\nu, g_n; x)| \geqslant C\zeta(a_n)\, |a_n^{1/2} p_n(x)|^\nu\, x^{-\langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle} (\log x)^{\#}. \tag{5.13}$$

First, by (2.5), (2.6), and (2.8) we see that for $0 < \kappa < 1/2$ small enough,

$$\| W(x)\, p_n(x) \|_{L_p(\kappa a_n,\, 2a_n)} \sim \| W(x)\, p_n(x) \|_{L_p(R)}. \tag{5.14}$$

Therefore, by (5.4), (5.5), (5.13), (5.14), and (2.9), we have

$$\begin{aligned} C &\geqslant \|(1+|x|)^{-\Delta}\, W^\nu(x)\, L_n(\nu, g_n; x)\|_{L_p(\kappa a_n,\, 2a_n)} \\ &\geqslant C\zeta(a_n)\, a_n^{\nu/2} a_n^{-(\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle)} (\log n)^{\#} \\ &\quad \times \| W(x)\, p_n(x) \|^\nu_{L_{p\nu}(\kappa a_n,\, 2a_n)} \\ &\geqslant C\zeta(a_n)\, a_n^{1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle)} (\log n)^{\#}\, n^{(\nu/6)\{1-4)(p\nu)\}} \qquad \text{(by } 4/\nu < p\text{)} \\ &\geqslant C\zeta(a_n)\, a_n^{1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle) + (\nu m/6)\{1-4/(p\nu)\}} (\log n)^{\#} \qquad \text{(by (2.2))}. \end{aligned}$$

Therefore, we have

$$C\{\log(1+n)\}^{1/(2p)} \geqslant a_n^{1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle) + (m/6)(\nu - 4/p)} (\log n)^{\#}. \tag{5.15}$$

Consequently, if $\alpha + (\nu-1)\, m/6 = 1$, then we see that

$$1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle) + (m/6)(\nu - 4/p) < 0$$

(recall the definition of $(\log n)^{\#}$), therefore we have (1.17). If $\alpha + (\nu-1)\, m/6 \neq 1$, then (5.15) implies that

$$C\{\log(1+n)\}^{1/(2p)} \geqslant a_n^{1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle) + (m/6)(\nu - 4/p)}.$$

Therefore, we have

$$1/p - (\Delta + \langle\!\langle \alpha + (\nu-1)\, m/6 \rangle\!\rangle) + (m/6)(\nu - 4/p) \leqslant 0.$$

Thus, we have (1.18). Consequently, the theorem follows. ∎

Proof of Corollary 7. Let $\nu = 3, 5, 7, \ldots$, and let $\alpha \geqslant 1$. Furthermore, we assume that $(m/6)(\nu - 4/p) > 1$ for $\nu > 3$, or if $\nu = 3$, then $p > 4/3$. Then, the condition (1.13) is equivalent to the condition (1.18). ∎

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